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Painlevé integrability and multi-dromion solutions of the 2+1 dimensional AKNS system

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Abstract. The Painlevé integrability of the 2+1 dimensional AKNS system is proved. Using the standard truncated Painlevé expansion which corresponds to a special Bäcklund transformation, some special types of the localized excitations like the solitoff solutions, multi-dromion solutions and multi-ring soliton solutions are obtained.

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1 Introduction

In (1+1)-dimensions, the AKNS (Ablowitz-Kaup-Newell-Segur) system is one of the most important physical models. In (2+1)-dimensions, there exist several possible integrable extensions for the AKNS system, for example the DS (Davey-Stewartson) type system [1] and the Breaking soliton type system [2]. The (1+1)-dimensional AKNS system can be obtained from the usual symmetry constraint of the KP (Kadomtsev-Petviashvili) equation. Recently, we have obtained another type of (2+1)-dimensional AKNS system:

$$ip_t + p_{xx} + pu_x = 0, \quad (1)$$
$$-iq_t + q_{xx} + qu_x = 0, \quad (2)$$
$$u_y + pq = 0 \quad (3)$$

from the inner parameter dependent symmetry constraints of the KP equation [3]. When we take $y = x$, the system (1–3) is reduced to the usual (1+1)-dimensional AKNS system. If q is selected as the complex conjugation, then the system (1–3) can be considered as the asymmetric part of the DS system.

One of the most powerful method to prove the integrability of a model is the so-called Painlevé analysis developed by WTC (Weiss-Tabor-Canvela) [4]. Furthermore, the Painlevé analysis can also be used to find some exact solutions no matter whether the model is integrable or not. If one needs only to prove the Painlevé property of a model one may use the Kruskal's simplification for the WTC approach [5]. If one hopes to find some more information from the model, one has to use the original WTC approach or some extended forms [6–8]. In this paper, we combine the standard WTC approach with the Kruskal's simplification to simplify the proof of the Painlevé integrability and to obtain some exact solutions at the same time. In Section 2 of this paper, we study the Painlevé integrability of the (2+1)-dimensional AKNS system. In Section 3, we use the truncated Painlevé expansion to obtain a special type of solution of the model with some arbitrary functions. In the last section, some special types of localized solutions like the multi-solitoff solutions, multi-dromion solutions and the multi-ring soliton solutions are discussed.

2 Painlevé integrability of the (2+1)-dimensional AKNS system

According to the standard WTC approach, if the (2+1)-dimensional AKNS system is Painlevé integrable, then all the possible solutions of the model can be written as

$$p = \sum_{k=0}^{\infty} p_k f^{k+\alpha},\ q = \sum_{k=0}^{\infty} q_k f^{k+\beta},\ u = \sum_{k=0}^{\infty} u_k f^{k+\gamma}, \quad (4)$$

with sufficient arbitrary functions among $p_k,\ q_k,\ u_k$ in addition to f, where $\alpha,\ \beta$ and γ should all be the negative integer. In other words, the solutions of the model are single valued about an arbitrary singularity manifold.

To fix the constants $\alpha,\ \beta$ and γ, one use the standard leading order analysis. By substituting $p \sim p_0 f^{\alpha}$, $q \sim q_0 f^{\beta}$

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and $u \sim u_0 f^\gamma$ into (1–3) and comparing the leading order terms for $f \sim 0$, we get only one possible branch

$$\alpha = \beta = \gamma = -1, \quad (5)$$

$$u_0 = 2f_x,\ q_0 p_0 = 2f_x f_y. \quad (6)$$

Substituting (4) with (5) and (6) into (1–3) and vanishing all the coefficients of the powers f^k, we can obtain the recursion relations to determine the functions $u_k,\ p_k$, and q_k:

$$\begin{pmatrix} k(k-3)f_x^2 & 0 & (k-1)f_x p_0 \\ 0 & k(k-3)f_x^2 & (k-1)f_x q_0 \\ q_0 & p_0 & (k-1)f_y \end{pmatrix} \begin{pmatrix} p_k \\ q_k \\ u_k \end{pmatrix} = \begin{pmatrix} P_k \\ Q_k \\ U_k \end{pmatrix}, \quad (7)$$

where

$$P_k = -i(p_{k-2,t} + (k-2)p_{k-1} f_t) - p_{k-2,xx} - 2(k-2)p_{k-1,x} f_x - (k-2)p_{k-1} f_{xx} - \sum_{i=0}^{k-1} u_{i,x} p_{k-i-1} - \sum_{i=0}^{k-1} (i-1) f_x u_i p_{k-i}, \quad (8)$$

$$Q_k = i(q_{k-2,t} + (k-2)q_{k-1} f_t) - q_{k-2,xx} - 2(k-2)q_{k-1,x} f_x - (k-2)q_{k-1} f_{xx} - \sum_{i=0}^{k-1} u_{i,x} q_{k-i-1} - \sum_{i=1}^{k-1} (i-1) f_x u_i q_{k-i}, \quad (9)$$

$$U_k = -u_{k-1,y} - \sum_{i=1}^{k-1} p_i q_{k-i} \quad (10)$$

with $p_k = q_k = u_k = 0$ for $k < 0$. From the matrix equation (7), it is easy to know that if the determinant

$$\Delta \equiv \det A = (k+1)k(k-1)(k-3)(k-4) f_x^4 f_y, \quad (11)$$

of the coefficient matrix

$$A \equiv \begin{pmatrix} k(k-3)f_x^2 & 0 & (k-1)f_x p_0 \\ 0 & k(k-3)f_x^2 & (k-1)f_x q_0 \\ q_0 & p_0 & (k-1)f_y \end{pmatrix}, \quad (12)$$

is not equal to zero, then the functions $p_k,\ q_k$ and u_k can be obtained from (7) uniquely. When

$$k = -1,\ 0,\ 1,\ 3,\ 4, \quad (13)$$

resonances occur. The resonance at $k = -1$ corresponds to that the singularity manifold f being arbitrary. If the model is Painlevé integrable, we require four resonance conditions at $k = 0,\ 1,\ 3,\ 4$ which are satisfied identically such that the other four arbitrary functions among $p_k,\ q_k,\ u_k$ can be introduced into the general series expansion (4). From the leading order analysis result (6) we know that the resonance at $k = 0$ is satisfied identically and one of p_0 and q_0 is arbitrary. For $k = 1$, we can obtain

$$p_1 = -\frac{ip_0 f_t + 2p_{0x} f_x - p_0 f_{xx}}{f_x u_0}, \quad (14)$$

$$q_1 = \frac{iq_0 f_t - 2q_{0x} f_x + q_0 f_{xx}}{f_x u_0}, \quad (15)$$

and the resonance condition

$$2u_{0y} f_x^2 - (2q_{0x} f_x - q_0 f_{xx}) p_0 - (2p_{0x} f_x - p_0 f_{xx}) q_0 = 0. \quad (16)$$

It is straightforward to see that the resonance condition (16) is satisfied identically because of (6). For $k = 2$, (7) gives us

$$p_2 = \frac{1}{6f_x^2 q_0}(p_0(iq_{0t} - q_1 u_{0x} - q_{0xx}) + 2q_0(ip_{0t} + p_{0xx} + p_1 u_{0x}) + u_0(f_y u_{1x} - p_1 q_1 f_x - f_x u_{1y})), \quad (17)$$

$$q_2 = \frac{1}{6f_x^2 p_0}(-q_0(ip_{0t} + p_1 u_{0x} + p_{0xx}) + 2p_0(-iq_{0t} + q_{0xx} + q_1 u_{0x}) + u_0(f_y u_{1x} - p_1 q_1 f_x - f_x u_{1y})), \quad (18)$$

$$u_2 = \frac{1}{6f_x^2 f_y}((-2u_{1x} p_0 - ip_{0t} - p_{0xx} - u_{0x} p_1) q_0 + (-q_{0xx} + iq_{0t} - u_{0x} q_1) p_0 - 2f_x^2 u_{1y} - 2f_x^2 q_1 p_1); \quad (19)$$

For $k = 3$, the component form of (7) reads

$$(2f_x u_3 + u_{2x}) p_0 + (f_{xx} + u_{0x} + if_t) p_2 + (u_{1x} + f_x u_2) p_1 + 2p_{2x} f_x + ip_{1t} + p_{1xx} = 0, \quad (20)$$

$$(2f_x u_3 + u_{2x}) q_0 + (f_{xx} + u_{0x} - if_t) q_2 + (u_{1x} + f_x u_2) q_1 + 2q_{2x} f_x - iq_{1t} + q_{1xx} = 0, \quad (21)$$

$$u_{2y} + 2u_3 f_y + q_2 p_1 + q_1 p_2 + q_3 p_0 + q_0 p_3 = 0. \quad (22)$$

Now, to verify the remaining resonance conditions at $k = 3$ and $k = 4$, we use the Kruskal's simplification, *i.e.*:

$$f = x + \psi(y,t),\ p_k = p_k(y,t),\ q_k = q_k(y,t),\ u_k = u_k(y,t). \quad (23)$$

One can use this simplification without loss of the generality to prove the Painlevé integrability, where $\psi(y,t)$ is an arbitrary function of $\{y,\ t\}$. Under the simplification (23), the known results become

$$u_0 = 2,\ q_0 p_0 = 2\psi_y,\ p_1 = -\frac{1}{2} i p_0 \psi_t,\ q_1 = \frac{1}{2} i q_0 \psi_t, \quad (24)$$

$$p_2 = \frac{1}{12\psi_y}(2i\psi_y p_{0t} - p_0(2u_{1y} + \psi_t^2 \psi_y - 2i\psi_{ty})), \quad (25)$$

$$q_2 = -\frac{1}{12\psi_y}(2i\psi_y q_{0t} + q_0(2u_{1y} + \psi_t^2 \psi_y + 2i\psi_{ty})), \quad (26)$$

$$u_2 = \frac{1}{6f_y}(iq_{0t}p_0 - ip_{0t}q_0 - 2u_{1y} - 2q_1 p_1). \quad (27)$$

Using (24–27), we can obtain

$$u_3 = -\frac{1}{4}\psi_{tt} \quad (28)$$

from (20) while (21) becomes a resonance condition which is really satisfied identically because of (24–27) and (28). p_3 and q_3 are related by (22) which can be simplified to

$$-\psi_{tt}\psi_y - \psi_{ty}\psi_t + 2u_{2y} + 2q_0 p_3 + 2q_3 p_0 = 0 \quad (29)$$

that means one of p_3 and q_3 is arbitrary. For $k = 4$, the resonance condition reads

$$\begin{aligned} & u_{3y} + q_3 p_1 + q_1 p_3 + q_2 p_2 \\ & \quad + \frac{1}{4}(-u_2 q_2 + iq_{2t} + 2iq_3\psi_t - 2u_3 q_1)p_0 \\ & \quad + \frac{1}{4}(-u_2 p_2 - ip_{2t} - 2ip_3\psi_t - 2u_3 p_1)q_0 = 0, \end{aligned} \quad (30)$$

while p_4 and q_4 are expressed as

$$p_4 = -\frac{1}{4}(3u_4 p_0 + ip_{2t} + 2ip_3\psi_t + 2u_3 p_1 + u_2 p_2), \quad (31)$$

$$q_4 = \frac{1}{4}(iq_{2t} - 3u_4 q_0 + 2iq_3\psi_t - 2u_3 q_1 - u_2 q_2) \quad (32)$$

for arbitrary u_4. After substituting (24–29), (31) and (32) into (30) one can see that the resonance condition (30) is also satisfied identically. Hence the (2+1)-dimensional AKNS system (1–3) is Painlevé integrable.

3 General solution from the truncated Painlevé expansion

It is known that the Painlevé analysis can also be used to obtain other interesting properties [4]. In this paper, we use the truncated Painlevé expansion to obtain some interesting exact solutions. If we take

$$q_k = p_k = u_k = 0, \qquad k \geq 2, \quad (33)$$

(4) with (5) becomes a truncated expansion

$$p = \frac{p_0}{f} + p_1,\ q = \frac{q_0}{f} + q_1,\ u = \frac{u_0}{f} + u_1. \quad (34)$$

Substituting (34) into (1–3) or substituting (33) directly to (14–22), we know that $\{p_1,\ q_1,\ u_1\}$ is also a solution of the (2+1)-dimensional AKNS system. In other words, the truncated expansion (24) is really a Bäcklund transformation. For simplicity later, we select the seed solution

$$p_1 = 0,\ q_1 = 0,\ u_1 = g(x,t) \equiv g, \quad (35)$$

where g is an arbitrary function of $\{x,\ t\}$. Under the special selection of (35), all the remained equations to solve the functions p_0 and f are simplified to

$$f_{xy} = 0, \quad (36)$$

$$ip_{0t} + p_{0xx} + p_0 g_x = 0, \quad (37)$$

$$-2f_x p_{0x} - if_t p_0 + f_{xx} p_0 = 0, \quad (38)$$

$$-p_0 p_{0xy} + p_{0y} p_{0x} = 0, \quad (39)$$

with

$$q_0 = \frac{2f_y f_x}{p_0},\ u_0 = 2f_x. \quad (40)$$

If we fix $g(x,t)$ as

$$g(x,t) = -\int (F_{1xx} - F_{1x}^2)\mathrm{d}x - iF_{0t}x + F_3, \quad (41)$$

then the only solution of (36–39) is

$$f = f_1(x,t) + f_2(y), \qquad p_0 = \exp(F_1(x) + F_2(y) + F_0(t)) \quad (42)$$

where $f_2(y) \equiv f_2,\ F_1(x) \equiv F_1,\ F_2(y) \equiv F_2,\ F_0(t) \equiv F_0$ and $F_3(t) \equiv F_3$ are all arbitrary functions of the indicated variables while $f_1(x,t) \equiv f_1$ is related to F_1 by

$$2f_{1x}F_{1x} + if_{1t} - f_{1xx} = 0. \quad (43)$$

Now it is interest to study the structure of the potential pq. From (34, 35, 40) and (42), we have

$$pq = \frac{2f_{2y}f_{1x}}{(f_1 + f_2)^2}. \quad (44)$$

4 Special examples

Because of the arbitrariness of the functions F_1 and f_2, the structure of pq shown by (44) is quite rich. Some special examples are listed here:

4.1 Multi-solitoff solutions

If we take f_1 as t independent then we can consider $f_1(x)$ is arbitrary while F_1 is determined by (43). Furthermore, we can take

$$f_2 = \sum_{i=1}^{N} b_i \exp(l_i y + y_{0i}), \qquad f_1 = \sum_{i=1}^{M} c_i \exp(k_i x + x_{0i}), \quad (45)$$

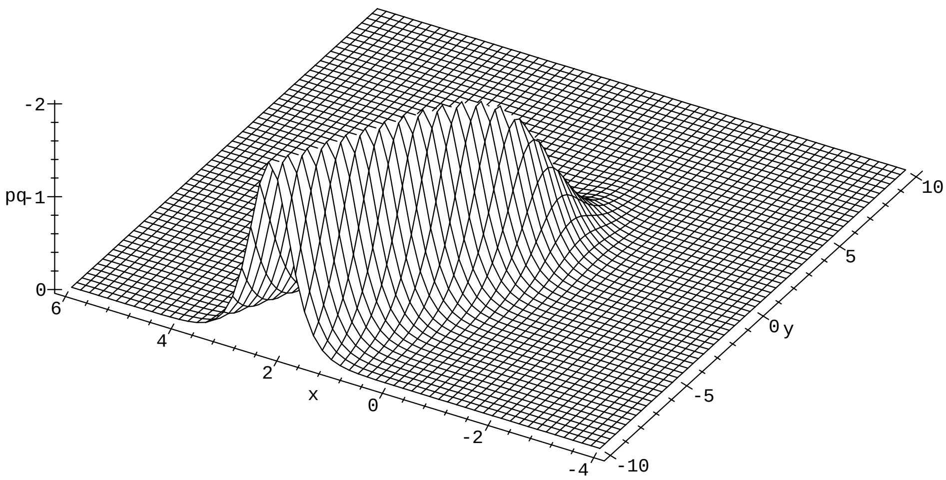

Fig. 1. A solitoff solution driven by two perpendicular line solitons with the selection (46).

where x_{0i}, y_{0i}, b_i, c_i, k_i and l_i are arbitrary constants and M and N are arbitrary positive integers. Under the selection (45), we obtain the first type of the multiple straight line solitons. Figure 1 shows a special structure of a two static resonant straight line solitons for the quantity pq with

$$f_1 = 1 + \exp(4x + 1), \; f_2 = \exp(-y). \tag{46}$$

From Figure 1, one can see that two straight line solitons become only one half straight line soliton because of the resonance effect. This type of half straight line soliton is called solitoff solution in the literature [9]. In reference [9] solitoff solutions obtained for other types of models like the asymmetric Nizhnik-Novikov-Veselov (NNV) equation and the (2+1)-dimensional dispersive long wave equation. It seems that the solitoff solutions are also a quite general phenomena in (2+1)-dimensional models.

4.2 Multi-dromion solutions

If we select the functions f_1 and f_2 themselves as some kinds of multiple straight line solitons parallel to y and x axes respectively, then (27) denotes some types of multiple dromion solutions. For instance, if we take

$$f_1 = C + \sum_{i=1}^{M} a_i \tanh(k_i x + x_{i0}), \tag{47}$$

$$f_2 = \sum_{i=1}^{N} b_i \tanh(K_i y + y_{i0}), \tag{48}$$

where K_i, k_i, x_{i0}, y_{i0}, a_i, b_i and $C > \sum_{i=1}^{M} |a_i| + \sum_{i=1}^{N} |b_i|$ are all arbitrary constants, then the quantity pq shown by (44) denotes an $M \times N$ dromion solution. The $M \times N$ dromions are located at the $M \times N$ "lattice" and the $M \times N$ lattice points are just the cross points of $M + N$ "ghost" straight line solitons. Figure 2 shows a six dromion solution (44) with

$$f_1 = 3 + \frac{1}{2}\tanh\left(\frac{3}{2}x\right) + \frac{1}{2}\tanh\left(\frac{2}{3}x + \frac{9}{2}\right) + \frac{1}{2}\tanh(2x - 12), \tag{49}$$

$$f_2 = \frac{1}{2}\tanh(y) + \frac{1}{2}\tanh(2y - 10). \tag{50}$$

The multiple dromion solutions have been found for many other types of models [10,11]. For instance, the dromion solution of the DS equation is found by Fokas and Santini [10]. In reference [10], the dromion solutions are constructed by the summations of the exponential functions. The dromion functions may have quite rich structures because we can use summations of any types of localized functions as well as rapidly divergent functions for f_1 and f_2 of (44).

4.3 Saddle-type multiple ring soliton solutions

In high dimensions, in addition to the point-like (soliton solutions with one or more peaks), there may be some other types of localized coherent solutions. In particular, some different types of single ring (a closed curve) soliton solutions have been found in some (2+1)-dimensions, for example, the single plateau-type, basin-type and bowl-type ring solitons [12] for the (2+1)-dimensional sine-Gordon equation [13] and the single saddle-type of ring soliton solutions for the NNV equation, ANNV equation and the dispersive long wave equations [14].

From the general solution (44), we can see that the saddle type of ring soliton solution also exists for the (2+1)-dimensional AKNS system by selecting the arbitrary functions appropriately. Figure 3 shows a two saddle type ring

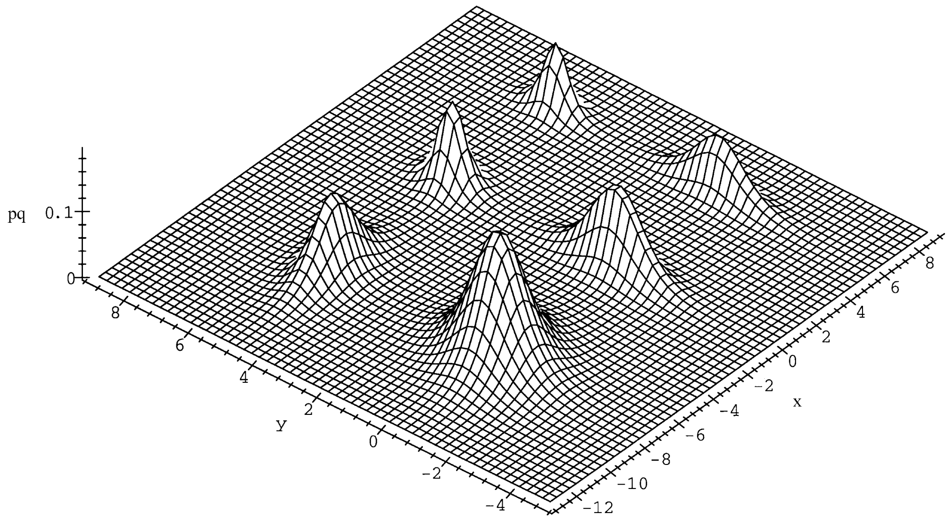

Fig. 2. A six dromion solution related to the selections (49) and (50).

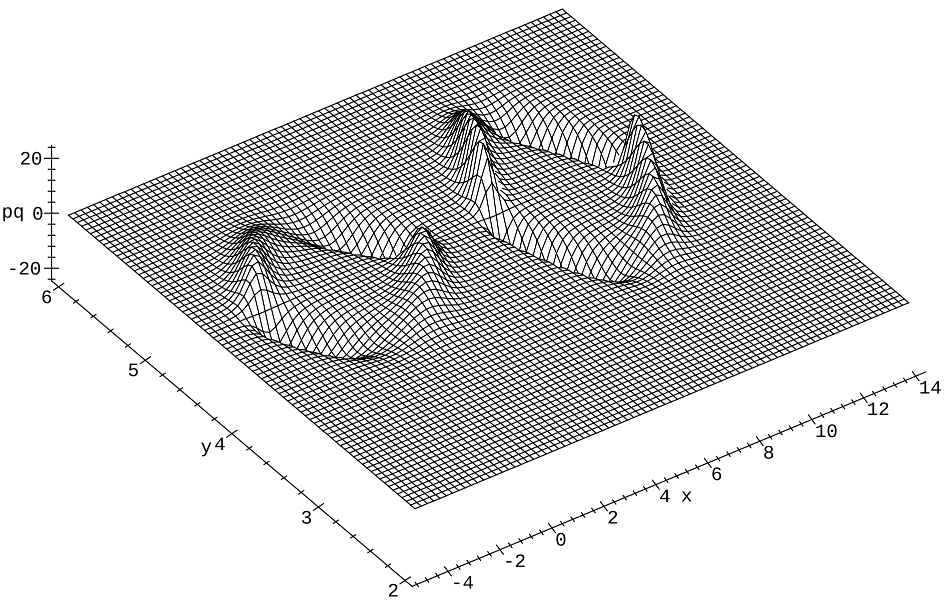

Fig. 3. A special two saddle type ring soliton solution (44) with (51) and (52).

soliton structure for the quantity pq shown by (44) with the selection

$$f_1 = \frac{1}{10}\exp\left(-\left(\frac{4x}{5}\right)^2 + 25\right) + \exp\left(-\frac{6}{5}(x - \frac{17}{2})^2 + 25\right), \tag{51}$$

$$f_2 = \frac{1}{10}\exp(y^2) + \exp\left(\left(y - \frac{17}{2}\right)^2\right) \cdot \tag{52}$$

In summary, for the (2+1)-dimensional Painlevé integrable AKNS system, a quite general solution with some arbitrary functions of x, y and t respectively can be obtained by using the truncated Painlevé expansion. By selecting the arbitrary functions appropriately, we may obtain many new types of multi-soliton solutions because of the existence of some arbitrary functions. The multiple localized excitation (44) may be solitoffs, dromions and saddle type ring solitons etc, and there is still more in the rich soliton structure in high dimensions worthy of study.

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